Informational Approach To Multiparametrical Quantum Estimation: Superresolving Imaging With Higher-Order Correlations

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Here we present an informational approach to multiparametrical quantum state/process estimation under the condition of parametrical locality, when the result of a particular measurement is dependent only on a particular limited subset of parameters. Such measurements are rather common for systems consisting of components well separated in coordinate or phase space. For example, near-field imaging [1] or data-pattern tomography [2] fall into this category. Our approach can be developed both for linear and nonlinear estimation problems, and allows for a drastic simplification of the parameter inference procedure. The complexity of the developed method can be linear on the total number of parameters.

We consider the general measurement model, $p_k = F_k(\theta_1, \ldots, \theta_M)$, where θ_j are parameters in question. We consider the Fisher information matrix: $F_{mn} = \sum_{\forall j} \frac{1}{p_j} \frac{\partial p_j}{\partial \theta_m} \frac{\partial p_j}{\partial \theta_n}$, where we assume $\sum_{\forall j} p_j = 1$. For the unbiased estimate, the Cramer-Rao bound connects the elements of the inverse Fisher matrix with the variance of the estimators, $\Delta^2(\theta_j) = [F^{-1}]_{jj}/N$, where N gives the total number of events described by the probabilities p_k. We would call the measurements strictly parametrically local if there exists such $l < M$ that $\frac{\partial p_j}{\partial \theta_m} = 0$ for $|m-j| > l$. In this case the Fisher matrix would be l-banded. For the inverse, F^{-1}, of the l-banded matrix F one can always find such a $n \times l$-banded matrix A that the distance between F^{-1} and A decreases as $\frac{1}{\lambda_{min}} |(\lambda_{max} - \lambda_{min})/(\lambda_{max} + \lambda_{min})|^{n+1}$, where λ_{min} (λ_{max}) is the minimal (maximal) eigenvalue of F [3]. In a similar manner one can show that if the Fisher information matrix is only approximately banded, i.e. it has some small non-zero elements away from the main bands, one can still derive a similar bound and show that the inverse matrix is also approximately banded.

The constructed bounds for the inverse of the Fisher matrix imply that the variance of the estimator for θ_j is not affected by variations of parameters out of some vicinity of θ_j (e.g. θ_k with $|k - j| > nl$). It points to the possibility of developing a "sliding window" approach for the local parameter inference. To get an accurate estimate of the parameter θ_h one can fix a window size $J \geq nl$ and perform a local reconstruction by optimizing the distance between the measured frequencies f_k, $k \in [h - J, h + J]$, and the estimated probabilities $p_k(\{\theta_i\})$, $i \in [1, M]$, where the values of the parameters θ_i, $i \notin [h - J, h + J]$, outside the window, can be set equal to an arbitrary constant and excluded from the optimization. Banded structure of the inverse of the Fisher matrix ensures that the value of this constant has negligible effect on the reconstructed value of the parameter θ_h. Estimation can be performed for a sequence of h, thus, shifting the "estimation window" along the whole set of parameters (with possible update of the parameters lying on the border of the "window" on each step). Complexity of such a "sliding window" approach is linear on the number of shifts required to cover the whole set of parameters.

We apply the "sliding window" reconstruction procedure to a practically important example of quantum near-field imaging with high-order correlation functions. Measuring the high-order correlation functions is one of the ways to increase resolution of the reconstructed image, and to go beyond empirical "Abbe limit". The following model is considered. The imaging field described by the density matrix ρ impinges on the object in the object plane described by the transmission function $A(\vec{s})$, passes through the imaging system described the transfer function $h(\vec{s}, \vec{r})$ and goes to the detectors. The operator of the field amplitude, $E_0(\vec{s})$, at the object plane is connected to the operator of the field amplitude at the image plain, $E(\vec{r})$, in the following standard way: $E(\vec{r}) = \int d^2\vec{s}\, E_0(\vec{s}) A(\vec{s}) h(\vec{s}, \vec{r})$. The probability p_k of registering coincident detection events at a set of points $\vec{r}_1^{(k)}$, ..., $\vec{r}_n^{(k)}$ is determined by n-th order correlation function $G_k^{(n)} = \mathrm{Tr} \left\{ \left[\prod_j E(\vec{r}_j^{(k)})\right]^\dagger \left[\prod_j E(\vec{r}_j^{(k)})\right] \rho \right\}$. For an object, represented as a superposition of M pixels, $A(\vec{s}) = \sum_{j=1}^{M} d_j(\vec{s}) x_j$, where the functions $d_j(\vec{s})$ describe the ideal transmission through the jth pixel, and x_j is the value of the transmission of jth pixel, the detection probabilities p_k can be represented as polynomials on the transmissions.

The "sliding window" approach has been tested on measured $G^{(2)}$ images with twin-photons, generated by spontaneous parametric down-conversion, and measured $G^{(2)}$, $G^{(3)}$ and $G^{(4)}$ images with quasi-thermal light. Fig. 1 shows the reconstruction result for an image of three slits with the pixel size 5 times smaller than the classical "Abbe limit", shown by a horizontal bar (red one in the right panel).

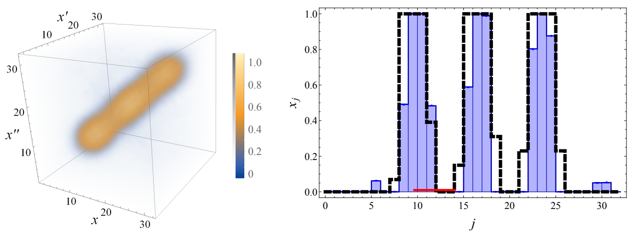

Figure 1: Measured $G^{(3)}(x, x', x'')$ image of 3 slits (left) and the reconstruction result (right panel; bars). Black line shows the pixel representation of the expected object.

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