New variance uncertainty relations for several observables

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The uncertainty relations for an arbitrary set of N observables were derived for the first time by Robertson [1], and many their special cases and generalizations were given in the review [2]. Unfortunately, such inequalities are too complicated in the most general form, because they contain, in addition to N variances of the observables and $N(N-1)/2$ mean values of commutators, numerous combinations of $N(N-1)/2$ covariances. I show a generalization of Robertson's scheme, which allows one to get rid of all covariances for $N = 3$ and $N = 4$. These "magic numbers" are related to the existence of three Pauli matrices and four Dirac matrices. An example for $N = 3$ is

$$X_{11}X_{22}X_{33} \geq \frac{4}{9}\left(X_{11}Y_{23}^2 + X_{22}Y_{13}^2 + X_{33}Y_{12}^2\right),$$

where $X_{jj} \equiv \langle z_j^2\rangle - \langle z_j\rangle^2$ and $Y_{jk} = \langle[z_j, z_k]\rangle/(2i)$. In particular, for the angular momentum components one obtains the inequality (where $L_j \equiv \langle\hat{L}_j\rangle$)

$$L_{xx}L_{yy}L_{zz} \geq \hbar^2\left(L_{xx}L_x^2 + L_{yy}L_y^2 + L_{zz}L_z^2\right)/9.$$

The well known inequality $L_{xx}L_{yy} \geq (\hbar^2/4)L_z^2$ becomes useless for the states with $L_z = 0$. A more informative inequality in such cases is $\Delta L_x \Delta L_y \geq \hbar^2\left|L_x L_y/\left(4L_{zz}\right)\right|$. The simplest inequalities for the *sums* of variances are

$$X_{11} + X_{22} + X_{33} \geq 2\left[Y_{12}^2 + Y_{23}^2 + Y_{13}^2\right]^{1/2},$$

$$\left(\sum_{k=1}^{4} X_{kk}\right)^2 \geq 4\sum_{j<k} Y_{jk}^2 + 8\left|Y_{12}Y_{34} + Y_{23}Y_{14} + Y_{31}Y_{24}\right|.$$

It is a challenge to generalize these nice inequalities for N arbitrary operators.

It is known that the minimal value of the product of variances of two non-commuting observables A and B is bigger than the standard Heisenberg-Robertson limit $\langle[\hat{A},\hat{B}]\rangle|^2/4$, if some constraints on the statistical properties of the quantum system are available, such as the correlation coefficient betwee two observables [3, 4, 5] or the degree of purity of the quantum system [2, 6]. Now I show how the uncertainty product is increased due to non-zero correlations between the selected two-variable system and some extra variables describing the "external world". For example, if $z_1 = x$, $z_2 = p_x$, $z_3 = y$, and there are correlations between these observables due to some kinds of entanglement, then $\Delta x \Delta p_x \geq \sqrt{(\hbar/2)^2 + (X_{12} - B)^2} + |B|$, where $B = X_{13}X_{23}/X_{33}$ and X_{jk} is the standard covariance between variables z_j and z_k. The equality is achieved for some specific pure Gaussian two-mode states. Inequalities

$$\Delta z_1 \Delta z_2 \geq 8|Y_{13}Y_{23}|/(9X_{33}),$$

$$\Delta z_1 \Delta z_2 \geq |Y_{23}Y_{14}|/(\Delta z_3 \Delta z_4)$$

hold if $Y_{12} = 0$. They show that even if the mean value of the commutator between two operators is zero, nonetheless, the product of variances of the corresponding observables must be nonzero, if these observables are parts of some extended system.

The proofs of the results can be found in Refs. [7, 8].

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